Programming and Computational Logic
A Motivational Introduction

The following people have contributed to this course material:

Manuel Hermenegildo (editor), Technical University of Madrid, Spain and University of New Mexico, USA; *Francisco Bueno, Manuel Carro, Pedro López, and Daniel Cabeza,* Technical University of Madrid, Spain; *María José García de la Banda,* Monash University, Australia; *David H. D. Warren,* University of Bristol, U.K.; *Ulrich Neumerkel,* Technical University of Vienna, Austria; *Michael Codish,* Ben Gurion University, Israel

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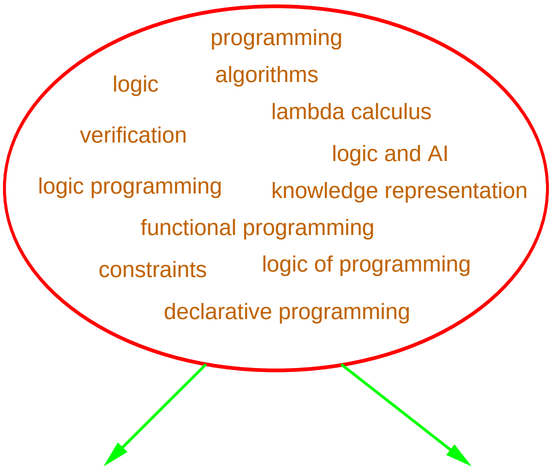
Course General Topic

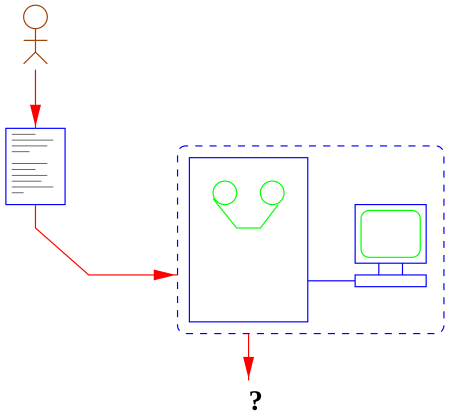
Computational Logic



Logic of Computation
program verification
proving properties

Declarative Programming
direct use of logic
as a programming tool

The Program Correctness Problem



- Conventional models of using computers – not easy to determine correctness!
    - Has become a very important issue, not just in safety-critical apps.
    - Components with assured quality, being able to give a warranty, ...
    - Being able to run untrusted code, certificate carrying code, ...

A Simple Imperative Program

- Example:

```
#include <stdio.h>
main() {
  int Number, Square;
  Number = 0;
       while(Number <= 5)
         { Square = Number * Number;
           printf("%d\n",Square);
           Number = Number + 1; } }
```

- Is it correct? With respect to what?
- A suitable formalism:
    - to provide *specifications* (describe problems), and
    - to reason about the *correctness of programs* (their *implementation*).

is needed.

Natural Language

"Compute the squares of the natural numbers which are less or equal than 5."

Ideal at first sight, but:

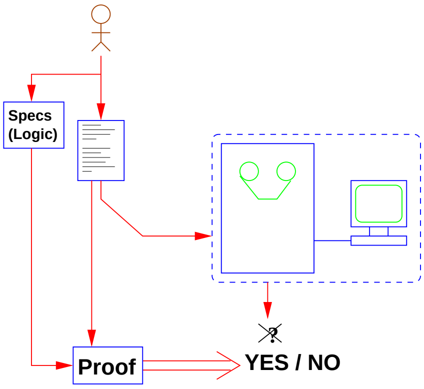
- verbose
- vague
- ambiguous
- needs context (assumed information)
- ...

Philosophers and Mathematicians already pointed this out a long time ago...

Logic

- A means of clarifying / formalizing the human thought process
- Logic for example tells us that (classical logic)
  Aristotle likes cookies, and
  Plato is a friend of anyone who likes cookies
  imply that
  Plato is a friend of Aristotle
- Symbolic logic:
  A shorthand for classical logic – plus many useful results:
  $a_1 : likes(aristotle, cookies)$
  $a_2 : \forall X\ likes(X, cookies) \rightarrow friend(plato, X)$
  $t_1 : friend(plato, aristotle)$
  $T[a_1, a_2] \vdash t_1$
- But, can logic be used:
  - To represent the problem (specifications)?
  - *Even perhaps to solve the problem?*

Using Logic



Specs (Logic)

Proof

YES / NO

- For expressing specifications and reasoning about the correctness of programs we need:
  - ◇ Specification languages (assertions), modeling, ...
  - ◇ Program semantics (models, axiomatic, fixpoint, ...).
  - ◇ Proofs: program *verification* (and debugging, equivalence, ...).

Generating Squares: A Specification (I)

Numbers —we will use “Peano” representation for simplicity:

0 → 0 1 → s(0) 2 → s(s(0)) 3 → s(s(s(0))) . . .

- Defining the natural numbers:
  $nat(0) \land nat(s(0)) \land nat(s(s(0))) \land \ldots$
- A better solution:
  $nat(0) \land \forall X\ (nat(X) \rightarrow nat(s(X)))$
- Order on the naturals:
  $\forall X\ (le(0, X)) \land$
  $\forall X \forall Y\ (le(X, Y) \rightarrow le(s(X), s(Y))$
- Addition of naturals:
  $\forall X\ (nat(X) \rightarrow add(0, X, X)) \land$
  $\forall X \forall Y \forall Z\ (add(X, Y, Z) \rightarrow add(s(X), Y, s(Z)))$

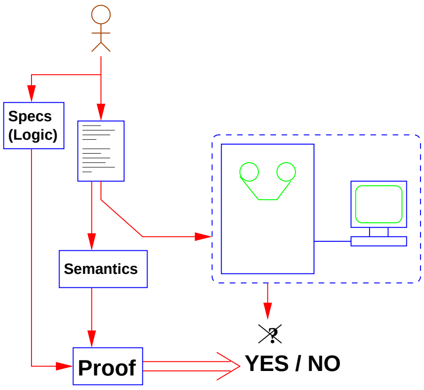
Generating Squares: A Specification (II)

- Multiplication of naturals:
  $\forall X\ (nat(X) \rightarrow mult(0, X, 0)) \wedge$
  $\forall X \forall Y \forall Z \forall W\ (mult(X, Y, W) \wedge add(W, Y, Z) \rightarrow mult(s(X), Y, Z))$
- Squares of the naturals:
  $\forall X \forall Y\ (nat(X) \wedge nat(Y)\ \wedge mult(X, X, Y) \rightarrow nat\_square(X, Y))$

We can now write a *specification* of the (imperative) program, i.e., conditions that we want the program to meet:

- *Precondition:*
  empty.
- *Postcondition:*
  $\forall X (output(X) \leftarrow (\exists Y\ nat(Y) \wedge le(Y, s(s(s(s(s(0)))))) \wedge nat\_square(Y, X)))$

Use of Logic



Specs (Logic)

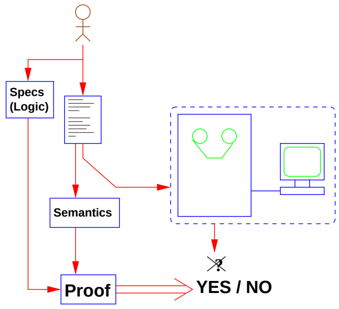
Semantics

Proof

YES / NO

- For expressing specifications and reasoning about the correctness of programs we need:
  - Specification languages (assertions), modeling, ...
  - Program semantics (models, axiomatic, fixpoint, ...).
  - Proofs: program *verification* (and debugging, equivalence, ...).

Semantic Tasks



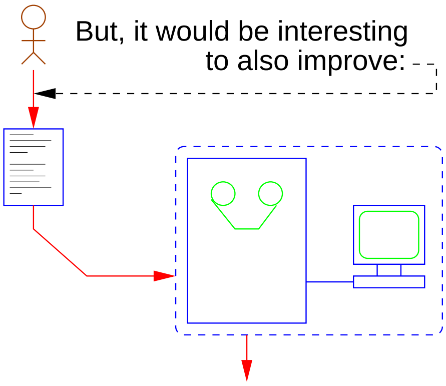
- Semantics:
  - A *semantics* associates a meaning (a mathematical object) to a program or program sentence.
- Semantic tasks:
  - Verification: proving that a program meets its specification.
  - Static debugging: finding where a program does not meet specifications.
  - Program equivalence: proving that two programs have the same semantics.
  - etc.

Styles of Semantics

- **Operational:**
  The meaning of program sentences is defined in terms of the steps (transformations from state to state) that computations may take during execution (derivations). Proofs by induction on derivations.
- **Axiomatic:**
  The meaning of program sentences is defined indirectly in terms some axioms and rules of some logic of program properties.
- **Denotational (fixpoint):**
  The meaning of program sentences is given abstractly as elements of some suitable mathematical structure (domain).
- **Model (declarative) semantics:**
  The meaning of programs is given as a minimal model (“logical meaning”) of the logic that the program is written in.

Alternative Use of Logic?

- So, logic allows us to *represent problems* (program specifications).



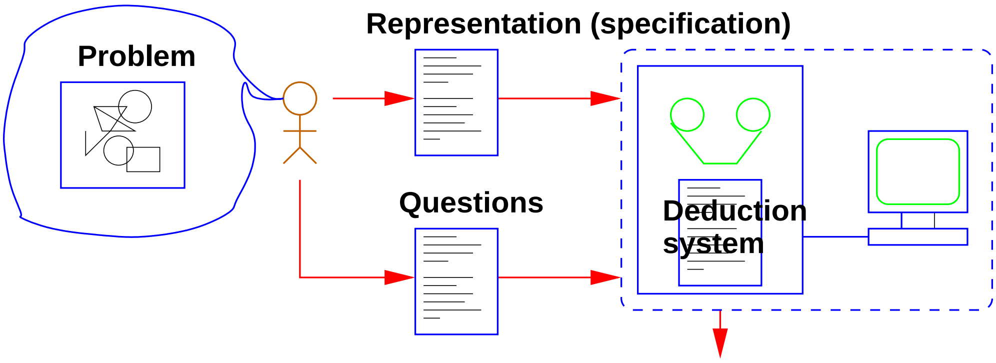
But, it would be interesting
to also improve:

i.e., the process of implementing solutions to problems.

- The importance of Programming Languages (and tools).
- Interesting question: can logic help here too?

From Representation/Specification to Computation

- Assuming the existence of a *mechanical proof method* (deduction procedure) *a new view of problem solving and computing is possible* [Greene]:
  - ◇ program once and for all the deduction procedure in the computer,
  - ◇ find a suitable *representation* for the problem (i.e., the *specification*),
  - ◇ then, to obtain solutions, ask questions and let deduction procedure do rest:



Problem

Representation (specification)

Questions

Deduction system

(Correct) Answers / Results

- No correctness proofs needed!

Computing With Our Previous Description / Specification

Query	**Answer**
$nat(s(0))$ **?**	(yes)
$\exists X\ add(s(0), s(s(0)), X)$ **?**	$X = s(s(s(0)))$
$\exists X\ add(s(0), X, s(s(s(0))))$ **?**	$X = s(s(0))$
$\exists X\ nat(X)$ **?**	$X = 0 \vee X = s(0) \vee X = s(s(0)) \vee \ldots$
$\exists X \exists Y\ add(X, Y, s(0))$ **?**	$(X = 0 \wedge Y = s(0)) \vee (X = s(0) \wedge Y = 0)$
$\exists X\ nat\_square(s(s(0)), X)$ **?**	$X = s(s(s(s(0))))$
$\exists X\ nat\_square(X, s(s(s(s(0)))))$ **?**	$X = s(s(0))$
$\exists X \exists Y\ nat\_square(X, Y)$ **?**	$(X = 0 \wedge Y = 0) \vee (X = s(0) \wedge Y = s(0)) \vee (X = s(s(0)) \wedge Y = s(s(s(s(0))))) \vee \ldots$
$\exists X output(X)$ **?**	$X = 0\ \vee\ X = s(0)\ \vee\ X = s(s(s(s(0))))\ \vee\ X = s^{9}(0)\ \vee\ X = s^{16}(0)\ \vee\ X = s^{25}(0)$

Which Logic?

- We have already argued the convenience of representing the problem in logic, but
  - which logic?
    - propositional
    - predicate calculus (first order)
    - higher-order logics
    - modal logics
    - λ-calculus, ...
  - which reasoning procedure?
    - natural deduction, classical methods
    - resolution
    - Prawitz/Bibel, tableaux
    - bottom-up fixpoint
    - rewriting
    - narrowing, ...

Issues

- We try to maximize expressive power.
- But one of the main issues is whether we have an **effective** reasoning procedure.
- It is important to understand the underlying properties and the theoretical limits!
- Example: propositions vs. first-order formulas.
  - ◇ Propositional logic:

    “spot is a dog” *p*
    “dogs have tail” *q*

    but how can we conclude that Spot has a tail?
  - ◇ Predicate logic extends the expressive power of propositional logic:

    $dog(spot)$
    $\forall X\, dog(X) \rightarrow has\_tail(X)$

    now, using deduction we can conclude:

    $has\_tail(spot)$

Comparison of Logics (I)

- Propositional logic:

  “spot is a dog” *p*
  \+ decidability/completeness
  \- limited expressive power
  \+ practical deduction mechanism

  → circuit design, “answer set” programming, ...
- Predicate logic: (first order)

  “spot is a dog” *dog(spot)*
  +/- decidability/completeness
  +/- good expressive power
  \+ practical deduction mechanism (e.g., **SLD-resolution**)

  → classical logic programming!

Comparison of Logics (II)

- Higher-order predicate logic:

  “There is a relationship for spot” *X(spot)*
  - decidability/completeness
  + good expressive power
  – practical deduction mechanism

  But interesting subsets → HO logic programming, functional-logic programming, ...

- Other logics: decidability? Expressive power? Practical deduction mechanism? Often (very useful) variants of previous ones:
  - ◇ Predicate logic + constraints (in place of unification)
    → constraint programming!
  - ◇ Propositional temporal logic, etc.

- Interesting case: λ-calculus

  + similar to predicate logic in results, allows higher order
  - does not support predicates (relations), only functions

→ functional programming!

Generating squares by SLD-Resolution – Logic Programming (I)

- We code the problem as definite (Horn) clauses:
  $nat(0)$
  $\neg nat(X) \vee nat(s(X))$
  $\neg nat(X) \vee add(0, X, X))$
  $\neg add(X, Y, Z) \vee add(s(X), Y, s(Z))$
  $\neg nat(X) \vee mult(0, X, 0)$
  $\neg mult(X, Y, W) \vee \neg add(W, Y, Z) \vee mult(s(X), Y, Z)$
  $\neg nat(X) \vee \neg nat(Y) \vee \neg mult(X, X, Y) \vee nat\_square(X, Y)$
- **Query:** $nat(s(0))$ **?**
- In order to refute: $\neg nat(s(0))$
- Resolution:
  $\neg nat(s(0))$ with $\neg nat(X) \vee nat(s(X))$ gives $\neg nat(0)$
  $\neg nat(0)$ with $nat(0)$ gives $\Box$
- Answer: (yes)

Generating squares by SLD-Resolution – Logic Programming (II)

$nat(0)$
$\neg nat(X) \vee nat(s(X))$
$\neg nat(X) \vee add(0, X, X))$
$\neg add(X, Y, Z) \vee add(s(X), Y, s(Z))$
$\neg nat(X) \vee mult(0, X, 0)$
$\neg mult(X, Y, W) \vee \neg add(W, Y, Z) \vee mult(s(X), Y, Z)$
$\neg nat(X) \vee \neg nat(Y) \vee \neg mult(X, X, Y) \vee nat\_square(X, Y)$

- **Query:** $\exists X \exists Y \ add(X, Y, s(0))$ **?**
- In order to refute: $\neg add(X, Y, s(0))$
- Resolution:
  $\neg add(X, Y, s(0))$ with $\neg nat(X) \vee add(0, X, X))$ gives $\neg nat(s(0))$
  $\neg nat(s(0))$ solved as before
- Answer: $X = 0, Y = s(0)$
- Alternative:
  $\neg add(X, Y, s(0))$ with $\neg add(X, Y, Z) \vee add(s(X), Y, s(Z))$ gives $\neg add(X, Y, 0)$

Generating Squares in a Practical Logic Programming System (I)

```
:- module(_,_,['bf/af']).

nat(0) <- .
nat(s(X)) <- nat(X).

le(0,_X) <- .
le(s(X),s(Y)) <- le(X,Y).

add(0,Y,Y) <- nat(Y).
add(s(X),Y,s(Z)) <- add(X,Y,Z).

mult(0,Y,0) <- nat(Y).
mult(s(X),Y,Z) <- add(W,Y,Z), mult(X,Y,W).

nat_square(X,Y) <- nat(X), nat(Y), mult(X,X,Y).

output(X) <- nat(Y), le(Y,s(s(s(s(0))))), nat_square(Y,X).
```

Generating Squares in a Practical Logic Programming System (II)

Query	Answer
?- `nat(s(0)).`	yes
?- `add(s(0),s(s(0)),X).`	`X = s(s(s(0)))`
?- `add(s(0),X,s(s(s(0)))).`	`X = s(s(0))`
?- `nat(X).`	`X = 0 ; X = s(0) ; X = s(s(0)) ; ...`
?- `add(X,Y,s(0)).`	`(X = 0 , Y=s(0)) ; (X = s(0) , Y = 0)`
?- `nat_square(s(s(0)), X).`	`X = s(s(s(s(0))))`
?- `nat_square(X,s(s(s(s(0))))).`	`X = s(s(0))`
?- `nat_square(X,Y).`	`(X = 0 , Y=0) ; (X = s(0) , Y=s(0)) ; (X = s(s(0)) , Y=s(s(s(s(0))))) ; ...`
?- `output(X).`	`X = 0 ; X = s(0) ; X = s(s(s(s(0)))) ; ...`