Computational Logic

Automated Deduction Fundamentals

Elements of First-Order Predicate Logic

First Order Language:

- An *alphabet* consists of the following classes of symbols:
    1. *variables* denoted by $X, Y, Z, Boo, ...,$ (infinite)
    2. *constants* denoted by $1, a, boo, john, ...,$
    3. *functors* denoted by $f, g, +, -, ..,$
    4. *predicate symbols* denoted by $p, q, dog, ...,$
    5. *connectives*, which are: $\neg$ (negation), $\vee$ (disjunction), $\wedge$ (conjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence),
    6. *quantifiers*, which are: $\exists$ (there exists) and $\forall$ (for all),
    7. *parentheses*, which are: (and) and the *comma*, that is: ",".
- Each functor and predicate symbol has a fixed *arity*, they are often represented in $Functor/Arity$ form, e.g. f/3.
- A constant can be seen as a functor of arity 0.
- Propositions are represented by a predicate symbol of arity 0.

Important: Notation Convention Used

(A bit different from standard notational conventions in logic, but good for compatibility with LP systems)

- Variables: start with a capital letter or a "_" (X, Y, _a, _1)
- Atoms, functors, predicate symbols: start with a lower case letter or are enclosed in ' ' (f, g, a, 1, x, y, z, 'X', '_1')

Terms and Atoms

We define by induction two classes of strings of symbols over a given alphabet.

- The class of *terms*:
  - a variable is a term,
  - a constant is a term,
  - if f is an n-ary functor and $t_1, ..., t_n$ are terms then $f(t_1, ..., t_n)$ is a term.
- The class of *atoms* (different from LP!):
  - a proposition is an atom,
  - if p is an n-ary pred. symbol and $t_1, ..., t_n$ are terms then $p(t_1, ..., t_n)$ is an atom,
  - $\mathbf{true}$ and $\mathbf{false}$ are atoms.
- The class of Well Formed Formulas (WFFs):
  - an atom is a WFF,
  - if F and G are WFFs then so are $\neg F, (F \vee G), (F \wedge G), (F \rightarrow G)$ and $(F \leftrightarrow G)$,
  - if F is a WFF and X is a variable then $\exists X\ F$ and $\forall X\ F$ are WFF.
- Literal: positive or negative (non-negated or negated) atom.

Examples

Examples of Terms

- Given:
  - constants: *a, b, c, 1, spot, john*...
  - functors: *f/1, g/3, h/2, +/3*...
  - variables: *X, L, Y*...
- Correct: *spot, f(john), f(X), +(1,2,3), +(X,Y,L), f(f(spot)), h(f(h(1,2)),L)*
- Incorrect: *spot(X), +(1,2), g, f(f(h))*

Examples of Literals

- Given the elements above and:
  - predicate symbols: *dog/1, p/2, q/0, r/0, barks/1*...
- Correct: *q, r, dog(spot), p(X,f(john))*...
- Incorrect: *q(X), barks(f), dog(barks(X))*

Examples (Contd.)

Examples of WFFs

- Given the elements above
- Correct: *q*, $q \rightarrow r$, $r \leftarrow q$, *dog(X)* $\leftarrow$ *barks(X)*, *dog(X)* , *p(X,Y)*, $\exists$ *X (dog(X)* $\wedge$ *barks(X)* $\wedge$ $\neg$ *q)*, $\exists$ *Y (dog(Y)* $\rightarrow$ *bark(Y))*
- Incorrect: $q \vee$, $\exists$ *p*

More about WFFs

- Allow us to represent knowledge and reason about it
  - ◇ Marcus was a man — *man(marcus)*
  - ◇ Marcus was a pompeian — *pompeian(marcus)*
  - ◇ All pompeians were romans — $\forall\ X\ pompeian(X) \rightarrow roman(X)$
  - ◇ Caesar was a ruler — *ruler(caesar)*
  - ◇ All romans were loyal to Caesar or they hated him

    $\forall X\ roman(X) \rightarrow loyalto(X,caesar) \vee hate(X,caesar)$
  - ◇ Everyone is loyal to someone — $\forall\ X\ \exists\ Y\ loyalto(X,Y)$
- We can now reason about this knowledge using standard deductive mechanisms.
- But there is in principle no guarantee that we will prove a given theorem.

Towards Efficient Automated Deduction

- *Automated deduction is search.*
- Complexity of search: directly dependent on branching factor at nodes (exponentially!).
- It is vital to cut down the branching factor:
  - ◇ Canonical representation of nodes (allows identifying identical nodes).
  - ◇ As few inference rules as possible.

Towards Efficient Automated Deduction (Contd.)

Clausal Form

- The complete set of logical operators ($\leftarrow, \wedge, \vee, \neg$,...) is redundant.
- A minimal (canonical) form would be interesting.
- It would be interesting to separate the quantifiers from the rest of the formula so that they did not need to be considered.
- It would also be nice if the formula were flat (i.e. no parenthesis).
- Conjunctive normal form has these properties [Davis 1960].

Deduction Mechanism

- A good example:
  Resolution – only two inference rules (*Resolution rule* and *Replacement rule*).

Classical Clausal Form: Conjunctive Normal Form

- General formulas are converted to:
  - Set of *Clauses*.
  - Clauses are in a logical conjunction.
  - A clause is a disjunction of the form. $literal_1 \lor literal_2 \lor \ldots \lor literal_n$
  - The $literal_i$ are negated or non-negated atoms.
  - All variables are implicitly universally quantified: i.e. if $X_1, ..., X_k$ are the variables that appear in a clause it represents the formula:
    $\forall X_1, ..., X_k \quad literal_1 \lor literal_2 \lor \ldots \lor literal_n$
- Any formula can be converted to clausal form automatically by:
  1. Converting to Prenex form.
  2. Converting to conjunctive normal form (conjunction of disjunctions).
  3. Converting to Skolem form (eliminating existential quantifiers).
  4. Eliminating universal quantifiers.
  5. Separating conjunctions into clauses.
- The *unsatisfiability* of a system is preserved.

Substitutions

- A **substitution** is a finite mapping from variables to terms, written as $\theta = \{X_1/t_1, ..., X_n/t_n\}$ where
  - ◇ the variables $X_1, ..., X_n$ are different,
  - ◇ for $i = 1, ..., n$ $X_i \neg \equiv t_i$.
- A pair X_i/t_i is called a **binding**.
- $domain(\theta) = \{X_1, .., X_n\}$ and $range(\theta) = vars(\{t_1, ..., t_n\})$.
- If $range(\theta) = \emptyset$ then θ is called **ground**.
- If θ is a bijective mapping from variables to variables then θ is called a **renaming**.
- Examples:
  - ◇ $\theta_1 = \{X/f(A), Y/X, Z/h(b, Y), W/a\}$
  - ◇ $\theta_2 = \{X/a, Y/a, Z/h(b, c), W/f(d)\}$ (ground)
  - ◇ $\theta_3 = \{X/A, Y/B, Z/C, W/D\}$ (renaming)

Substitutions (Contd.)

- Substitutions operate on *expressions*, i.e. a term, a sequence of literals or a clause, denoted by E.
- The application of θ to E (denoted $E\theta$) is obtained by *simultaneously* replacing each occurrence in E of X_i by t_i, $X_i/t_i \in \theta$.
- The resulting expression $E\theta$ is called an *instance* of E.
- If θ is a renaming then $E\theta$ is called a *variant* of E.
- Example:
  $\theta_1 = \{X/f(A), Y/X, Z/h(b, Y), W/a\}$
  $p(X, Y, X)\ \theta_1 = p(f(A), X, f(A))$

Composition of Substitutions

- Given $\theta = \{X_1/t_1, ..., X_n/t_n\}$ and $\eta = \{Y_1/s_1, ..., Y_m/s_m\}$ their *composition* $\theta\eta$ is defined by removing from the set

$$\{X_1/t_1\eta, ..., X_n/t_n\eta, Y_1/s_1, ..., Y_m/s_m\}$$

  those pairs $X_i/t_i\eta$ for which $X_i \equiv t_i\eta$, as well as those pairs Y_i/s_i for which $Y_i \in \{X_1, ..., X_n\}$.

- Example: if $\theta = \{X/3, Y/f(X,1)\}$ and $\eta = \{X/4\}$ then $\theta\eta = \{X/3, Y/f(4,1)\}$.

- For all substitutions θ, η *and* γ and an expression E

  i) $(E\theta)\eta \equiv E(\theta\eta)$

  ii) $(\theta\eta)\gamma = \theta(\eta\gamma)$.

- θ is **more general** than η if for some γ we have $\eta = \theta\gamma$.

- Example: $\theta = \{X/f(Y)\}$ more general than $\eta = \{X/f(h(G))\}$

Unifiers

- If $A\theta \equiv B\theta$, then
  - θ is called a *unifier* of A and B
  - A and B are *unifiable*
- A unifier θ of A and B is called a *most general unifier* (*mgu*) if it is *more general* than any other unifier of A and B.
- If two atoms are unifiable then they have a most general unifier.
- θ is **idempotent** if $\theta\theta = \theta$.
- A unifier θ of A and B is **relevant** if all variables appearing either in $domain(\theta)$ or in $range(\theta)$, also appear in A or B.
- If two atoms are unifiable then they have an mgu which is idempotent and relevant.
- An mgu is unique up to renaming.

Unification Algorithm

- Non-deterministically choose from the set of equations an equation of a form below and perform the associated action.
  1. $f(s_1, ..., s_n) = f(t_1, ..., t_n) \rightarrow$ replace by $s_1 = t_1, ..., s_n = t_n$
  2. $f(s_1, ..., s_n) = g(t_1, ..., t_m)$ where $f \not\equiv g \rightarrow$ halt with failure
  3. $X = X \rightarrow$ delete the equation
  4. $t = X$ where t is not a variable $\rightarrow$ replace by the equation $X = t$
  5. $X = t$ where $X \not\equiv t$ and X has another occurrence in the set of equations $\rightarrow$

     5.1 if X appears in t then halt with failure

     5.2 otherwise apply $\{X/t\}$ to every other equation
- Consider the set of equations $\{f(X) = f(f(Z)), g(a, Y) = g(a, X)\}$:
  - ◇ (1) produces $\{X = f(Z), g(a, Y) = g(a, X)\}$
  - ◇ then (1) Yields $\{X = f(Z), a = a, Y = X\}$
  - ◇ (3) produces $\{X = f(Z), Y = X\}$
  - ◇ now only (5) can be applied, giving $\{X = f(Z), Y = f(Z)\}$
  - ◇ No step can be applied, the algorithm successfully terminates.

Unification Algorithm revisited

- Let A and B be two formulas:

  1. $\theta = \epsilon$
  2. while $A\theta \neq B\theta$:

     2.1 find leftmost symbol in $A\theta$ s.t. the corresponding symbol in $B\theta$ is different

     2.2 let t_A and t_B be the terms in $A\theta$ and $B\theta$ starting with those symbols

     (a) if neither t_A nor t_B are variables or one is a variable occurring in the other $\rightarrow$ halt with failure

     (b) otherwise, let t_A be a variable $\rightarrow$ the new θ is the result of $\theta\{t_A/t_B\}$
  3. end with θ being an m.g.u. of A and B

Unification Algorithm revisited (Contd.)

- Example: $A = p(X, X)$ $B = p(f(A), f(B))$

θ	$A\theta$	$B\theta$	Element
ϵ	$p(X, X)$	$p(f(A), f(B))$	$\{X/f(A)\}$
$\{X/f(A)\}$	$p(f(A), f(A))$	$p(f(A), f(B))$	$\{A/B\}$
$\{X/f(B), A/B\}$	$p(f(B), f(B))$	$p(f(B), f(B))$	

- Example: $A = p(X, f(Y))$ $B = p(Z, X)$

θ	$A\theta$	$B\theta$	Element
ϵ	$p(X, f(Y))$	$p(Z, X)$	$\{X/Z\}$
$\{X/Z\}$	$p(Z, f(Y))$	$p(Z, Z)$	$\{Z/f(Y)\}$
$\{X/f(Y), Z/f(Y)\}$	$p(f(Y), f(Y))$	$p(f(Y), f(Y))$	

Resolution with Variables

- It is a *formal system* with:
  - A first order language with the following formulas:
    - Clauses: without repetition, and without an order among their literals.
    - The empty clause $\Box$.
  - An empty set of axioms.
  - Two inference rules: *resolution* and *replacement*.

Resolution with Variables (Contd.)

- Resolution:

$$\frac{\begin{array}{l} r_1\text{: } A \vee F_1 \vee \cdots \vee F_n \\ r_2\text{: } \neg B \vee G_1 \vee \cdots \vee G_m \end{array}}{((F_1 \vee \cdots \vee F_n)\sigma \vee G_1 \vee \cdots \vee G_m)\theta}$$

  where

  - A and B are unifiable with substitution θ
  - σ is a renaming s.t. $(A \vee F_1 \vee \cdots \vee F_n)\sigma$ and $\neg B \vee G_1 \vee \cdots \vee G_m$ have no variables in common
  - θ is the m.g.u. of $A\sigma$ and B

  The resulting clause is called the *resolvent* of r_1 and r_2.

- Replacement: $A \vee B \vee F_1 \vee \cdots \vee F_n \Rightarrow (A \vee F_1 \vee \cdots \vee F_n)\theta$ where
  - A and B are unifiable atoms
  - θ is the m.g.u. of A and B

Basic Properties

- Resolution is *correct* – i.e. all conclusions obtained using it are valid.
- There is no guarantee of directly deriving a given theorem.
- However, resolution (under certain assumptions) is refutation complete: if we have a set of clauses $K = [C_0, C_1, \ldots, C_n]$ and it is inconsistent then resolution will arrive at the empty clause $\Box$ in a finite number of steps.
- Therefore, a valid theorem (or a question that has an answer) is guaranteed to be provable by refutation. To prove "p" given $K_0 = [C_0, C_1, \ldots, C_n]$:
  1. Negate it ($\neg p$).
  2. Construct $K = [\neg p, C_0, C_1, \ldots, C_n]$.
  3. Apply resolution steps repeatedly to K.
- Furthermore, we can obtain answers by composing the substitutions along a path that leads to $\Box$ (very important for realizing Green's dream!).
- It is important to use a good method in applying the resolution steps – i.e. in building the resolution tree (or proof tree).
- Again, the main issue is to reduce the branching factor.

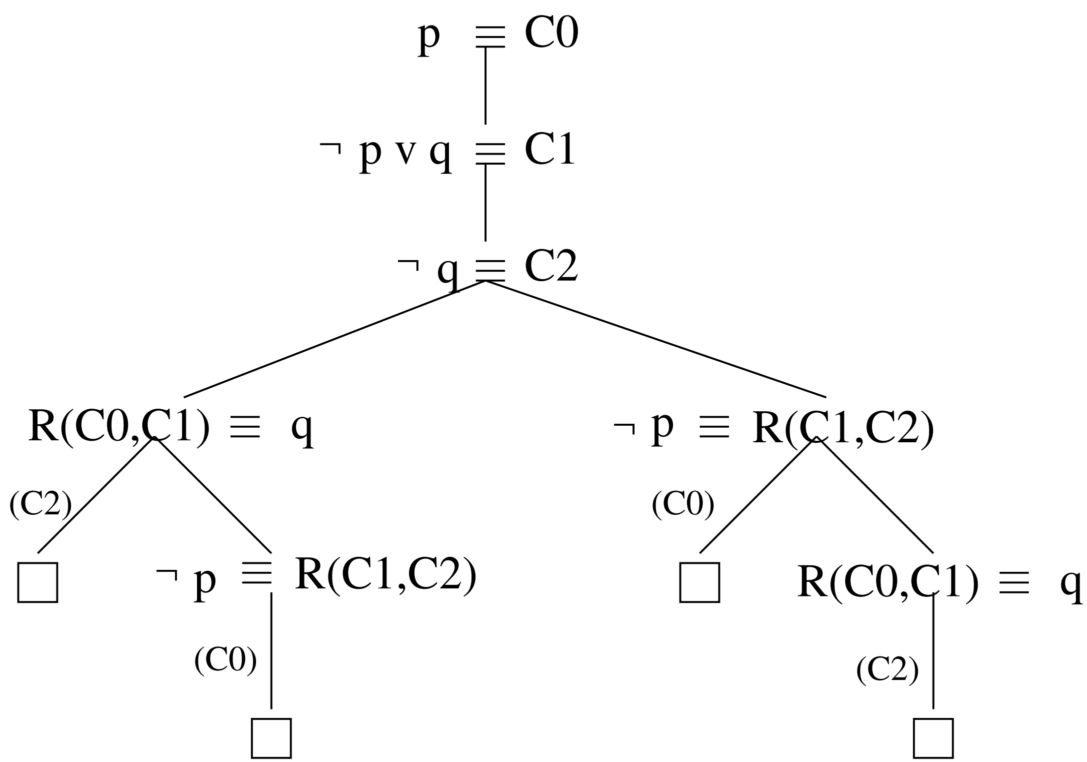
Proof Tree

- Given a set of clauses $K = \{C_0, C_1, \cdots, C_n\}$ the proof tree of K is a tree s.t. :
  - ◇ the root is C_0
  - ◇ the branch from the root starts with the nodes labeled with $C_0, C_1, \cdots, C_n$
  - ◇ the descendent nodes of C_n are labeled by clauses obtained from the parent clauses using resolution
  - ◇ a derivation in K is a branch of the proof tree of K
- The derivation $C_0 C_1 \cdots C_n F_0 \cdots F_m$ is denoted as $K, F_0 \cdots F_m$

Proof Tree (Contd.)

- Example: part of the proof tree for K, with:

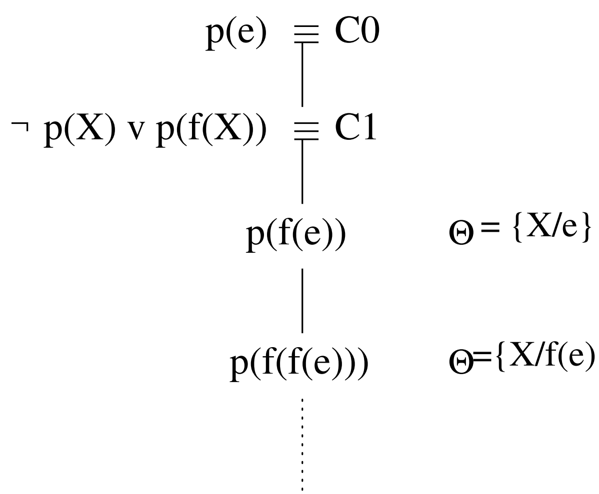
$$K = [\ p, \neg p \vee q, \neg\ q]$$



Characteristics of the Proof Tree

- It can be infinite:

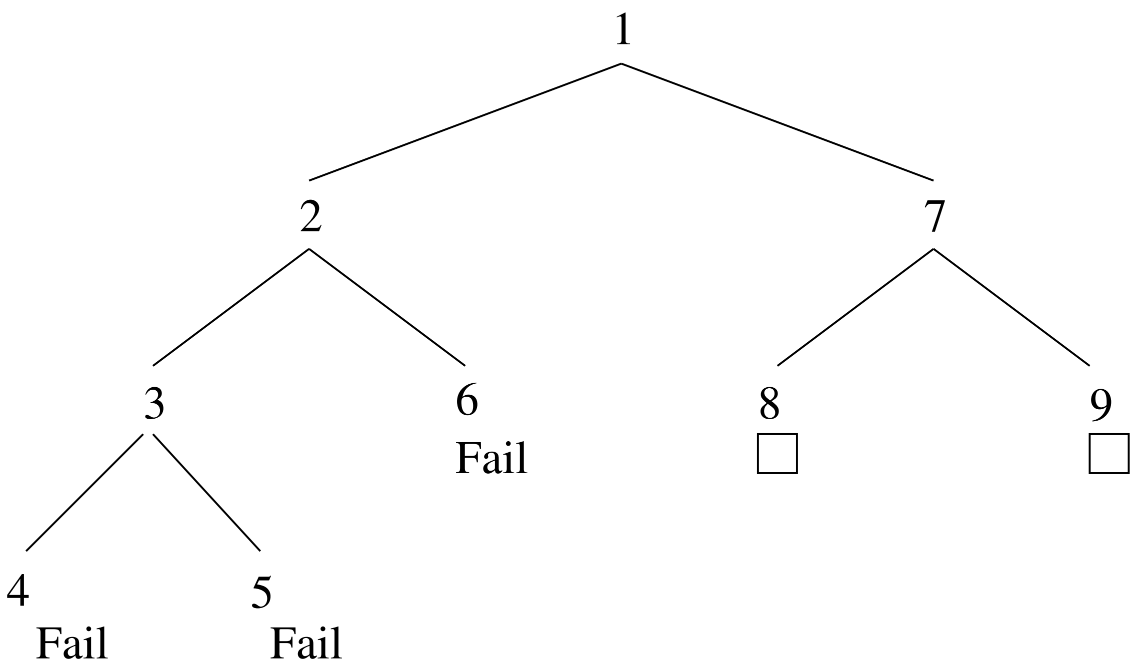
$K = [\ p(e), \neg p(X) \vee p(f(X))\ ]$



- Even if it is finite, it can be too large to be explored efficiently
- Aim: determine some criteria to limit the number of derivations and the way in which the tree is explored $\Rightarrow$ strategy
- Any strategy based on this tree is correct: if $\Box$ appears in a subtree of the proof tree of K, then $\Box$ can be derived from K and therefore K is unsatisfiable

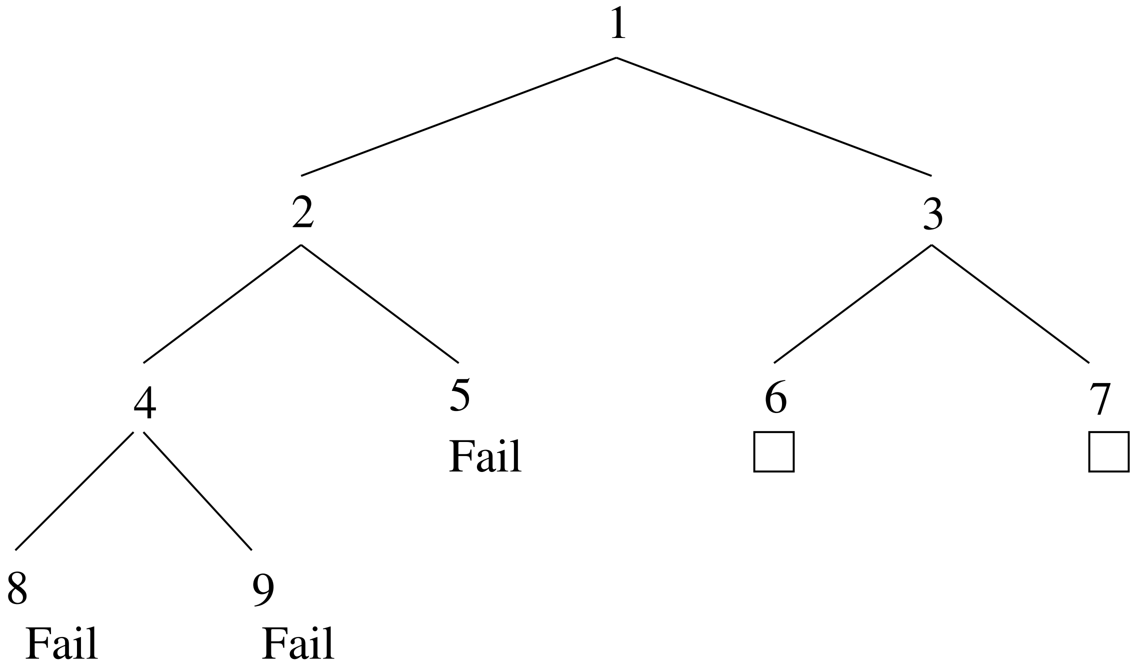
General Strategies

- **Depth-first with backtracking**: First descendant to the left; if failure or □ then backtrack



General Strategies (Contd.)

- **Breadth first**: all sons of all sibling nodes from left to right



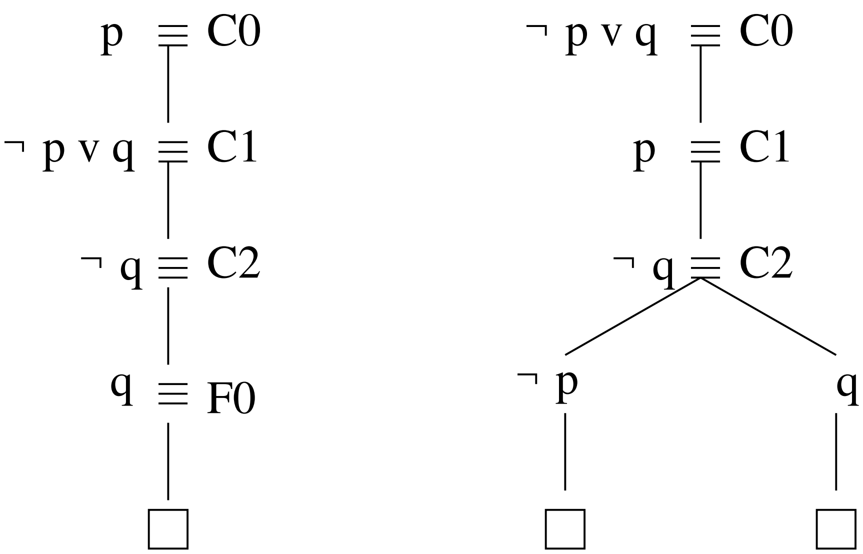
General Strategies (Contd.) (Contd.)

- **Iterative deepening**
  - ◇ Advance depth-first for a time.
  - ◇ After a certain depth, switch to another branch as in breadth-first.

- Completeness issues / possible types of branches:
  - ◇ Success (always finite)
  - ◇ Finite failure
  - ◇ Infinite failure (provably infinite branches)
  - ◇ Non-provably infinite branches

Linear Strategies

- Those which only explore linear derivations
- A derivation $K, F_0 \cdots F_m$ is linear if
  - ◇ F_0 is obtained by resolution or replacement using C_0
  - ◇ $F_i, i > 0$ is obtained by resolution or replacement using F_{i-1}
- Examples:

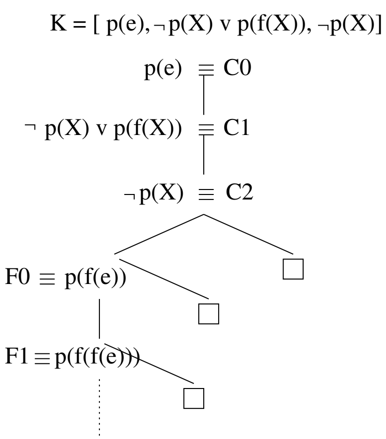


Characteristics of these Strategies

1 If $\Box$ can be derived from K by using resolution with variables, it can also be derived by linear resolution

2 Let K be $K' \cup \{C_0\}$ where K' is a satisfiable set of clauses, i.e. $\Box$ cannot be derived from K' by using resolution with variables. If $\Box$ can be derived from K by using resolution with variables it can also be derived by linear resolution with root C_0.

- From (1), if the strategy is breadth first, it is complete.
- From (2), if we want to prove that B is derived form K' then we can apply linear resolution to $K = K' \cup \{\neg B\}$.
- Depth first with backtracking is not complete:

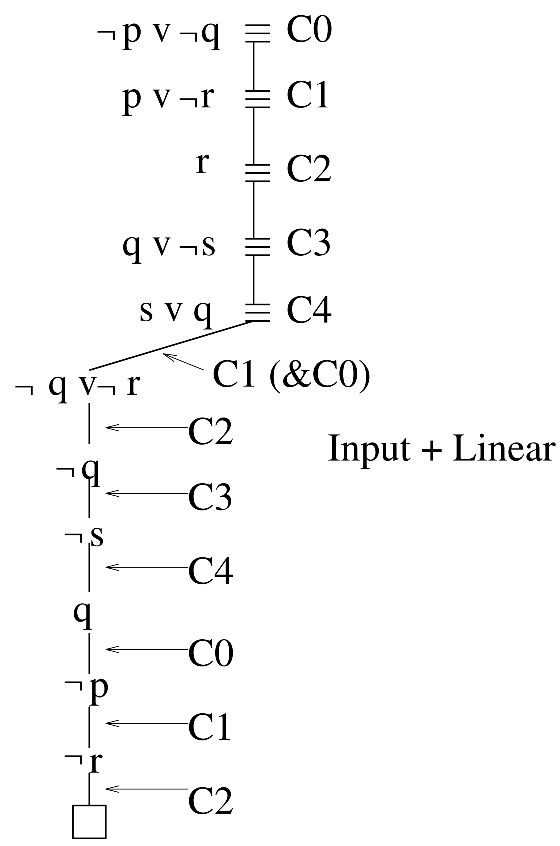


Input Strategies

- Those which only explore input derivations
- A derivation $K, F_0 \cdots F_m$ is input if
  - ◇ F_0 is obtained by resolution or replacement using C_0
  - ◇ $F_i, i > 0$ is obtained by resolution or replacement using at least a clause in K

- Example:

K = [¬p v¬ q, p v ¬r, r, q v¬ s, s v q]



Input Strategies

- In an input derivation, if F_{i-1} does not appear in any derivation of a successor clause, it can be eliminated from the derivation without changing the result
- If F_{i-1} appears in the derivation of $F_j, j > 1$, F_{i-1} can be allocated in position $j - 1$
- As a result, we can limit ourselves to linear input derivations without losing any input derivable clause
- Let K be $K' \cup \{C_0\}$ where $\square$ is derived by using resolution with variables, C_0 is a negative Horn clause and all clauses in K' are positive Horn clauses. There is an input derivation with root C_0 finishing in $\square$ and in which the replacement rule is not used (Hernschen 1974)
- A *Horn clause* is a clause in which at most one literal is positive:
  - ◇ it is *positive* if precisely one literal is positive
  - ◇ it is *negative* if all literals are negatives
- As a result, in those conditions, a breadth first input strategy is complete, and a depth first input strategy with backtracking is complete if the tree is finite.

Ordered Strategies

- We consider a new formal system in which:
  1. clauses are *ordered* sets
  2. ordered resolution of two clauses
     $A = p_1 \vee \cdots \vee p_n \; and \; B = q_1 \vee \cdots \vee q_m$
     where p_1 is a positive literal and q_1 is a negative literal is possible iff $\neg p_1$ and $\sigma(q_1)$ are unifiable (σ is a renaming, s.t. p_1 and $\sigma(q_1)$ have no variables in common)
  3. the resolvent of A and B is $\theta(p_2 \vee \cdots \vee p_n \vee \sigma(q_2 \vee \cdots \vee q_m))$ where θ is an m.g.u of $\neg p_1$ and $\sigma(q_1)$
- Let $K = K` \cup \{C_0\}$ be a set of clauses s.t. $\Box$ is derived by using resolution with variables, $C_0$ is a negative Horn clause and all clauses in $K`$ are positive Horn clauses with the positive literal in the first place. There is a sorted input derivation with root C_0 arriving at $\Box$.
- In this context a sorted linear input with:
  - ◇ breadth first: is complete
  - ◇ depth first with backtracking: is complete if the tree is finite